Int J Theor Phys (2008) 47: 1891–1910
DOI 10.1007/s10773-007-9634-5

SU(1, 1) Coherent States for the Generalized Two-Mode Time-Dependent Quadratic Hamiltonian System

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Received: 5 October 2007 / Accepted: 5 December 2007 / Published online: 18 December 2007

Abstract The SU(1, 1) coherent states, so-called Barut-Girardello coherent state and Perelomov coherent state, for the generalized two-mode time-dependent quadratic Hamiltonian system are investigated through SU(1, 1) Lie algebraic formulation. Two-mode Schrödinger cat states defined as an eigenstate of $\hat{K}_-^2$ are also studied. We applied our development to two-mode Caldirola-Kanai oscillator which is a typical example of the time-dependent quadratic Hamiltonian system. The time evolution of the quadrature distribution for the probability density in the coherent states are analyzed for the two-mode Caldirola-Kanai oscillator by plotting relevant figures.

Keywords SU(1, 1) coherent states · Time-dependent quadratic Hamiltonian system · Caldirola-Kanai oscillator

1 Introduction

For several decades, considerable attention has been paid to the investigation of quantum properties for the time-dependent quadratic Hamiltonian systems (TDQHS) [1–10]. Lots of fundamental systems in diverse fields of physics, including harmonic oscillators with a driving force, Caldirola-Kanai (CK) oscillator, and harmonic oscillator with time-dependent mass and/or frequency, are described by TDQHS. It has been turn out that TDQHS have many applications in non-relativistic description of quantum systems. For example, the evolution of cosmological constant [8], the propagation of light in time-varying media [9], and optical parametric amplifier [10] can be studied under the construction of time-dependent Hamiltonian. Besides the mathematical comprehension, the knowledge of

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exact quantum states for the TDHQS enables us to understand various physical information such as fluctuations, geometrical phase, quantum transition, and squeezing effect. In fact, the study of quantum features for TDQHS has been facilitated by the invention of a mathematical tool, namely the invariant operator [11] which is very useful in treating nonconservative quantum systems.

Although the original coherent states are introduced in harmonic oscillator system, the application of the coherent states is extended to wide range of physical systems, for instance, those which are described by SU(1, 1), SO(2, 1), and SU(2) Lie algebraic groups [6, 7, 12] or even nonlinear algebraic groups [13]. It is clear that the Lie algebraic approach is a powerful tool in investigating the quantum properties of dynamical systems. Therefore, the studies in many fields of physical systems, especially the non-classical properties of quantum optical systems including coherent and squeezed states have been fulfilled by employing SU(1, 1) Lie algebra [14–16]. In recent years, some novel applications of the group SU(1, 1) have been made in the field of quantum computation and tomography with nonlinear optics [17]. Among several classes of SU(1, 1) coherent states, we are interested in Barut-Girardello coherent states [18] and Perelomov coherent states [19] in this paper. The former may be generated by means of a process including the competition between nondegenerate parametric amplification and nondegenerate two-photon absorption, while the generation of the latter can be attained from the unitary evolution of two-mode number states driven by a nondegenerated parametric device [20]. It is noticed that SU(1, 1) coherent states reveal the nonclassical properties such as squeezing effect, nontrivial zero-point energy, violations of the Cauchy-Schwarz inequality, and fluctuation of uncertainty products [21, 22]. Since the SU(1, 1) coherent states for one-mode TDQHS have been already studied [6], the investigation of the SU(1, 1) coherent states in the present work will be focused on the generalized two-mode TDQHS through the formulation of SU(1, 1) Lie algebra.

This paper is organized as follows. In the following section, we realize SU(1, 1) Lie algebra by introducing SU(1, 1) generators for the two dimensional TDQHS. Two kind of SU(1, 1) coherent states, i.e., Barut and Girardello coherent states and Perelomov coherent states, are investigated in Sect. 3. Our development will be employed to the two-mode CK oscillator driven by external forces, which is one of the typical type of the TDQHS, in Sect. 4. Finally, the concluding remarks of this paper will be placed in the last section.

2 SU(1, 1) Lie Algebra for the Two-Mode TDQHS

In this section, we formulate SU(1, 1) Lie algebra for the two-mode quadratic Hamiltonian system whose parameters explicitly depend on time. The two-mode TDQHS can be represented by the sum of its two independent components as

$$\hat{H}_t(\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, t) = \hat{H}_1(\hat{x}_1, \hat{p}_1, t) + \hat{H}_2(\hat{x}_2, \hat{p}_2, t). \tag{2.1}$$

For the generalized TDQHS, $\hat{H}_j(\hat{x}_j, \hat{p}_j, t)$ (hereafter, all j is $j = 1, 2$) are given by

$$\begin{aligned}\hat{H}_j(\hat{x}_j, \hat{p}_j, t) = \frac{1}{2}[A_j(t)\hat{p}_j^2 + B_j(t)(\hat{x}_j\hat{p}_j + \hat{p}_j\hat{x}_j) + C_j(t)\hat{x}_j^2] \\ + D_j(t)\hat{x}_j + E_j(t)\hat{p}_j + F_j(t),\end{aligned} \tag{2.2}$$

where $A_j(t) - F_j(t)$ are time functions which are differentiable with respect to time. From Hamiltonian dynamics, we can easily confirm that the classical equation of motion for each mode can be written in the form

$$\frac{d^2x_j(t)}{dt^2}-\frac{\dot{A}_j}{A_j}\frac{dx_j(t)}{dt}+\left(A_jC_j+\frac{\dot{A}_jB_j}{A_j}-B_j^2-\dot{B}_j\right)x_j(t)=-\frac{\dot{A}_jE_j}{A_j}+B_jE_j-A_jD_j+\dot{E}_j. \tag{2.3}$$

The classical solutions of the above equations are the sum of complementary functions $x_{c,j}(t)$ and particular solutions $x_{p,j}(t)$ for each mode:

$$x_j(t)=x_{c,j}(t)+x_{p,j}(t). \tag{2.4}$$

Similarly, we can also denote complementary functions and particular solutions in momentum space as $p_{c,j}(t)$ and $p_{p,j}(t)$ so that the corresponding classical solutions have the form

$$p_j(t)=p_{c,j}(t)+p_{p,j}(t). \tag{2.5}$$

Many kinds of nonclassical phenomena in quantum systems may be studied in terms of SU(1, 1) Lie algebra. The two-mode bosonic realization of SU(1, 1) Lie algebra have been used to describe the nondegenerate parametric amplifier [23–25] while one mode realization used to present the degenerate parametric amplifier [26]. Based on the group theory, we introduce SU(1, 1) generators for the two-mode system such that

$$\hat{K}_0=\frac{1}{4\hbar}\sum_{j=1}^{2}\left\{\frac{\Omega_j}{s_j^2(t)}[\hat{x}_j-x_{p,j}(t)]^2\right.$$
$$\left.+\frac{1}{\Omega_j}\left[\frac{1}{A_j}\left(B_js_j(t)-\dot{s}_j(t)\right)[\hat{x}_j-x_{p,j}(t)]+s_j(t)[\hat{p}_j-p_{p,j}(t)]\right]^2\right\}, \tag{2.6}$$

$$\hat{K}_1=\frac{1}{2\hbar(\Omega_1\Omega_2)^{1/2}}\left\{\left[\frac{\Omega_1\Omega_2}{s_1(t)s_2(t)}-\frac{1}{A_1A_2}\left(B_1s_1(t)-\dot{s}_1(t)\right)\left(B_2s_2(t)-\dot{s}_2(t)\right)\right]\right.$$
$$\times[\hat{x}_1-x_{p,1}(t)][\hat{x}_2-x_{p,2}(t)]$$
$$-\frac{s_2(t)}{A_1}(B_1s_1(t)-\dot{s}_1(t))[\hat{x}_1-x_{p,1}(t)][\hat{p}_2-p_{p,2}(t)]$$
$$-\frac{s_1(t)}{A_2}(B_2s_2(t)-\dot{s}_2(t))[\hat{p}_1-p_{p,1}(t)][\hat{x}_2-x_{p,2}(t)]$$
$$\left.-s_1(t)s_2(t)[\hat{p}_1-p_{p,1}(t)][\hat{p}_2-p_{p,2}(t)]\right\}, \tag{2.7}$$

$$\hat{K}_2=-\frac{1}{2\hbar(\Omega_1\Omega_2)^{1/2}}\left\{\left[\frac{\Omega_1}{s_1(t)A_2}(B_2s_2(t)-\dot{s}_2(t))+\frac{\Omega_2}{s_2(t)A_1}(B_1s_1(t)-\dot{s}_1(t))\right]\right.$$
$$\times[\hat{x}_1-x_{p,1}(t)][\hat{x}_2-x_{p,2}(t)]$$
$$\left.+\frac{s_2(t)}{s_1(t)}\Omega_1[\hat{x}_1-x_{p,1}(t)][\hat{p}_2-p_{p,2}(t)]+\frac{s_1(t)}{s_2(t)}\Omega_2[\hat{p}_1-p_{p,1}(t)][\hat{x}_2-x_{p,2}(t)]\right\}, \tag{2.8}$$

where Ω_j are real positive constants and $s_j(t)$ are time-dependent variables that satisfy the following differential equations

$$\ddot{s}_j(t)-\frac{\dot{A}_j}{A_j}\dot{s}_j(t)+\left(A_jC_j+\frac{\dot{A}_jB_j}{A_j}-B_j^2-\dot{B}_j\right)s_j(t)-A_j^2\Omega_j^2\frac{1}{s_j^3(t)}=0. \tag{2.9}$$

These generators obey the commutation relations which are given by

$$[\hat{K}_0, \hat{K}_1] = i\hat{K}_2, \qquad [\hat{K}_0, \hat{K}_2] = -i\hat{K}_1, \qquad [\hat{K}_1, \hat{K}_2] = -i\hat{K}_0. \tag{2.10}$$

Note that the time-derivative of (2.6) ends up zero: $d\hat{K}_0/dt = 0$. It implies that $\hat{K}_0$ is a constant of motion which plays the role of invariant operator. The concept of constant of motion is important in studying quantum mechanics for the TDQHS [1] in addition to its evident usefulness in classical mechanics [27]. López et al. have deduced constant of motions for the various mechanical systems such as relativistic particle system with dissipation, nonrelativistic particle system driven by time-dependent force, time-dependent mass system, and position dependent mass system [28]. From (2.7) and (2.8), it is clear that two-mode generators $\hat{K}_1$ and $\hat{K}_2$ are not direct simple multiplications of two one-mode generators given in [6]. This is the reason why many researchers take a special interest in investigating the group SU(1, 1) for the two-mode systems separately from the one-mode ones.

Here, it is convenient to introduce the new generators, namely, raising operator and lowering operator correspond to SU(1, 1) Lie algebra:

$$\hat{K}_+ = \hat{K}_1 + i\hat{K}_2, \qquad \hat{K}_- = \hat{K}_1 - i\hat{K}_2. \tag{2.11}$$

From straightforward calculations, one may verify that operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_0$ satisfy the conventional commutation relations such that

$$[\hat{K}_-, \hat{K}_+] = 2\hat{K}_0, \qquad [\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm. \tag{2.12}$$

If we introduce another kind of lowering operators of the form [5]

$$\hat{a}_j = \sqrt{\frac{1}{2\hbar\Omega_j}}\left\{\left[\frac{\Omega_j}{s_j} + i\frac{B_j s_j - \dot{s}_j}{A_j}\right][\hat{x}_j - x_{p,j}(t)] + i s_j[\hat{p}_j - p_{p,j}(t)]\right\}, \tag{2.13}$$

and their Hermitian adjoints $\hat{a}_j^\dagger$ which are raising operators, $\hat{K}_0$ and $\hat{K}_\pm$ can be simply represented in the form

$$\hat{K}_0 = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1), \qquad \hat{K}_+ = \hat{a}_1^\dagger\hat{a}_2^\dagger, \qquad \hat{K}_- = \hat{a}_1\hat{a}_2. \tag{2.14}$$

Direct differentiations of $\hat{K}_\pm$ with respect to time yield

$$\frac{d\hat{K}_+}{dt} = i\sum_{j=1}^{2}\frac{\Omega_j A_j(t)}{s_j^2(t)}\hat{K}_+, \tag{2.15}$$

$$\frac{d\hat{K}_-}{dt} = -i\sum_{j=1}^{2}\frac{\Omega_j A_j(t)}{s_j^2(t)}\hat{K}_-, \tag{2.16}$$

We can easily identify that the solutions of the above equations are expressed as

$$\hat{K}_+(t) = \hat{K}_+(0)\exp\left(i\sum_{j=1}^{2}\int_0^t \frac{\Omega_j A_j(t')}{s_j^2(t')}dt'\right), \tag{2.17}$$

$$\hat{K}_-(t) = \hat{K}_-(0)\exp\left(-i\sum_{j=1}^{2}\int_0^t \frac{\Omega_j A_j(t')}{s_j^2(t')}dt'\right). \tag{2.18}$$

Therefore, we see that $\hat{K}_{\pm}(t)$ are represented in terms of constant magnitudes and time-dependent phases.

Under the basis vectors that are given by

$$|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle. \tag{2.19}$$

the eigenvalue equation of $\hat{K}_0$ can be expressed as

$$\hat{K}_0|n_1, n_2\rangle = \lambda_{n_1,n_2}|n_1, n_2\rangle. \tag{2.20}$$

Besides, the representations of the generators on the two-mode states are

$$\hat{K}_+|n_1, n_2\rangle = \sqrt{(n_1+1)(n_2+1)}|n_1+1, n_2+1\rangle, \tag{2.21}$$

$$\hat{K}_-|n_1, n_2\rangle = \sqrt{n_1 n_2}|n_1-1, n_2-1\rangle, \tag{2.22}$$

$$\hat{K}_0|n_1, n_2\rangle = \frac{1}{2}(n_1+n_2+1)|n_1, n_2\rangle, \tag{2.23}$$

$$\hat{K}_+\hat{K}_-|n_1, n_2\rangle = n_1 n_2|n_1, n_2\rangle. \tag{2.24}$$

From $\hat{K}_-\langle x_1, x_2|0,0\rangle = 0$, we can easily derive ground state of the system. And the nth order excited states can also be derived from

$$\langle x_1, x_2|n, n\rangle = \frac{1}{n!}\hat{K}_+^n\langle x_1, x_2|0,0\rangle = \langle x_1|n\rangle\langle x_2|n\rangle, \tag{2.25}$$

where $\langle x_j|n\rangle$ are the nth excited states in each mode [6]. Through these procedures, the two-mode eigenstates correspond to arbitrary combination of quantum numbers are obtained as

$$\begin{aligned}\langle x_1, x_2|n_1, n_2\rangle &= \left(\frac{(\Omega_1\Omega_2)^{1/2}}{\pi s_1 s_2 \hbar}\right)^{1/2} \frac{1}{\sqrt{2^{n_1+n_2}n_1!n_2!}} e^{ip_{p,1}(t)x_1/\hbar} e^{ip_{p,2}(t)x_2/\hbar} \\ &\quad \times H_{n_1}\left(\sqrt{\frac{\Omega_1}{s_1^2\hbar}}[x_1 - x_{p,1}(t)]\right) H_{n_2}\left(\sqrt{\frac{\Omega_2}{s_2^2\hbar}}[x_2 - x_{p,2}(t)]\right) \\ &\quad \times \exp\left\{-\sum_{j=1}^{2}\left[\frac{1}{2s_j\hbar}\left(\frac{\Omega_j}{s_j} + \frac{i}{A_j}(B_j s_j - \dot{s}_j)\right)[x_j - x_{p,j}(t)]^2\right]\right\}.\end{aligned} \tag{2.26}$$

where H_n is nth order Hermite polynomial.

It is well known that the wave functions of the system whose Hamiltonian is explicitly depend on time are the same as the eigenstates of the invariant operator, except for some time-dependent phase factor [1]. If we recall that $\hat{K}_0$ takes the role of the invariant operator, the wave functions in number state have the form

$$\langle x_1, x_2|\psi_{n_1,n_2}\rangle = \langle x_1, x_2|n_1, n_2\rangle \exp[i\theta_{n_1,n_2}(t)]. \tag{2.27}$$

By substituting (2.27) into Schrödinger equation, we can find the appropriate phases $\theta_{n_1,n_2}(t)$ as

$$\theta_{n_1,n_2}(t) = -\sum_{j=1}^{2}\left\{\left(n_j + \frac{1}{2}\right)\int_0^t \frac{A_j(t')\Omega_j}{s_j^2(t')}dt'\right.$$

$$+\frac{1}{\hbar}\int_0^t \left[\frac{1}{2}[A_j(t')p_{p,j}^2(t') - C_j(t')x_{p,j}^2(t')] + E_j(t')p_{p,j}(t') + F_j(t')\right]dt'\Bigg\}. \tag{2.28}$$

Thus, (2.27) with (2.26) and (2.28) are complete wave functions in number state. In the next section, we will investigate SU(1, 1) coherent states on the basis of the development presented in this section.

3 SU(1, 1) Coherent States

Coherent states are ubiquitous and important topic in the literature of mathematical physics. As a matter of fact, two-mode TDQHS permit plentiful coherent states, lots of which may be related to low-order Lie algebraic groups such as SU(1, 1) or SU(2) [29]. Numerous nonclassical phenomena such as phase coherence, quantum correlation, and squeezing of photons in quantum optical systems may be interpreted by means of the SU(1, 1) Lie algebra and the generalized coherent states associated with these Lie algebras [30]. Let us see two interesting analytic representation of SU(1, 1) coherent states, BG coherent state and Perelomov coherent state.

3.1 Barut-Girardello Coherent State

If we write the eigenvalue equation of $\hat{K}_-$ in the form

$$\hat{K}_-|\beta;q\rangle_{\mathrm{BG}} = \beta|\beta;q\rangle_{\mathrm{BG}}, \tag{3.1}$$

the BG coherent state is, actually, just the eigenstate $|\beta;q\rangle_{\mathrm{BG}}$. We can expand the BG coherent states in terms of the number states as

$$|\beta;q\rangle_{\mathrm{BG}} = \left[\frac{|\beta|^q}{I_q(2|\beta|)}\right]^{1/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!(q+n)!}}|n+q,n\rangle, \tag{3.2}$$

where I_q is a modified Bessel function of the first kind. It is known that the BG coherent state includes strong correlations with respect to the number states between the modes [29]. The density operator in this coherent state is given by

$$\hat{\rho}_{\mathrm{BG}} = |\beta;q\rangle_{\mathrm{BG}\ \mathrm{BG}}\langle\beta;q|. \tag{3.3}$$

The probability of finding $n+q$ quanta in mode 1 and n quanta in mode 2 in the BG coherent states is

$$P_n = \langle n+q,n|\hat{\rho}_{\mathrm{BG}}|n+q,n\rangle = \frac{1}{n!(q+n)!}\frac{|\beta|^{2n+q}}{I_q(2|\beta|)}. \tag{3.4}$$

Equation (3.2) may be used to investigate diverse quantum features such as probability density, variances, uncertainty product and quantum energy in the BG coherent state. The majority of these research requires the evaluation of relevant expectation values. After straightforward calculations, we can obtain the expectation values of the canonical variables in the form

$$ {}_{\mathrm{BG}}\langle\beta;q|\hat{x}_1|\beta;q\rangle_{\mathrm{BG}} = \sqrt{\frac{\hbar s_1^2}{2\Omega_1}}\frac{|\beta|^q}{I_q(2|\beta|)}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(\beta+\beta^*)}{n!(n+q)!\sqrt{n+1}} + x_{p,1}(t), \tag{3.5} $$

$$ {}_{\mathrm{BG}}\langle\beta;q|\hat{x}_2|\beta;q\rangle_{\mathrm{BG}} = \sqrt{\frac{\hbar s_2^2}{2\Omega_2}}\frac{|\beta|^q}{I_q(2|\beta|)}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(\beta+\beta^*)}{n!(n+q)!\sqrt{n+q+1}} + x_{p,2}(t), \tag{3.6} $$

$$ {}_{\mathrm{BG}}\langle\beta;q|\hat{p}_1|\beta;q\rangle_{\mathrm{BG}} = -\sqrt{\frac{\hbar}{2}}\frac{|\beta|^q}{I_q(2|\beta|)}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(Y_1\beta+Y_1^*\beta^*)}{n!(n+q)!\sqrt{n+1}} + p_{p,1}(t), \tag{3.7} $$

$$ {}_{\mathrm{BG}}\langle\beta;q|\hat{p}_2|\beta;q\rangle_{\mathrm{BG}} = -\sqrt{\frac{\hbar}{2}}\frac{|\beta|^q}{I_q(2|\beta|)}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(Y_2\beta+Y_2^*\beta^*)}{n!(n+q)!\sqrt{n+q+1}} + p_{p,2}(t), \tag{3.8} $$

$$ \begin{aligned} {}_{\mathrm{BG}}\langle\beta;q|\hat{x}_1^2|\beta;q\rangle_{\mathrm{BG}} &= \frac{|\beta|^q}{I_q(2|\beta|)}\left[\frac{\hbar s_1^2}{2\Omega_1}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(\beta^2+\beta^{*2})}{n!(n+q)!\sqrt{(n+1)(n+2)}}\right. \\ &\quad \left. + x_{p,1}(t)\sqrt{\frac{2\hbar s_1^2}{\Omega_1}}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(\beta+\beta^*)}{n!(n+q)!\sqrt{n+1}}\right] \\ &\quad + \frac{s_1^2\hbar}{2\Omega_1}\left[2|\beta|\frac{I_{q+1}(2|\beta|)}{I_q(2|\beta|)}+2q+1\right] + x_{p,1}^2(t), \end{aligned} \tag{3.9} $$

$$ \begin{aligned} {}_{\mathrm{BG}}\langle\beta;q|\hat{x}_2^2|\beta;q\rangle_{\mathrm{BG}} &= \frac{|\beta|^q}{I_q(2|\beta|)}\left[\frac{\hbar s_2^2}{2\Omega_2}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(\beta^2+\beta^{*2})}{n!(n+q)!\sqrt{(n+q+1)(n+q+2)}}\right. \\ &\quad \left. + x_{p,2}(t)\sqrt{\frac{2\hbar s_2^2}{\Omega_2}}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(\beta+\beta^*)}{n!(n+q)!\sqrt{n+q+1}}\right] \\ &\quad + \frac{s_2^2\hbar}{2\Omega_2}\left[2|\beta|\frac{I_{q+1}(2|\beta|)}{I_q(2|\beta|)}+1\right] + x_{p,2}^2(t), \end{aligned} \tag{3.10} $$

$$ \begin{aligned} {}_{\mathrm{BG}}\langle\beta;q|\hat{p}_1^2|\beta;q\rangle_{\mathrm{BG}} &= \frac{|\beta|^q}{I_q(2|\beta|)}\left[\frac{\hbar}{2}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(Y_1^2\beta^2+Y_1^{*2}\beta^{*2})}{n!(n+q)!\sqrt{(n+1)(n+2)}}\right. \\ &\quad \left. - \sqrt{2\hbar}p_{p,1}(t)\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(Y_1\beta+Y_1^*\beta^*)}{n!(n+q)!\sqrt{n+1}}\right] \\ &\quad + \frac{\hbar}{2}|Y_1|^2\left[2|\beta|\frac{I_{q+1}(2|\beta|)}{I_q(2|\beta|)}+2q+1\right] + p_{p,1}^2(t), \end{aligned} \tag{3.11} $$

$$ \begin{aligned} {}_{\mathrm{BG}}\langle\beta;q|\hat{p}_2^2|\beta;q\rangle_{\mathrm{BG}} &= \frac{|\beta|^q}{I_q(2|\beta|)}\left[\frac{\hbar}{2}\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(Y_2^2\beta^2+Y_2^{*2}\beta^{*2})}{n!(n+q)!\sqrt{(n+q+1)(n+q+2)}}\right. \\ &\quad \left. - \sqrt{2\hbar}p_{p,2}(t)\sum_{n=0}^{\infty}\frac{|\beta|^{2n}(Y_2\beta+Y_2^*\beta^*)}{n!(n+q)!\sqrt{n+q+1}}\right] \end{aligned} $$

$$+\frac{\hbar}{2}|Y_2|^2\left[2|\beta|\frac{I_{q+1}(2|\beta|)}{I_q(2|\beta|)}+1\right]+p_{p,2}^2(t), \tag{3.12}$$

where Y_j are

$$Y_j=\frac{B_j s_j-\dot{s}_j}{A_j\sqrt{\Omega_j}}+i\frac{\sqrt{\Omega_j}}{s_j}. \tag{3.13}$$

In the above evaluations we used the relation that (see page 5 of [31])

$$I_\nu(z)=\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{2n+\nu}\frac{1}{n!\Gamma(n+\nu+1)}. \tag{3.14}$$

The similar calculations of expectation values related to two-mode harmonic oscillator are found in [32]. However, they do not considered interference between two states correspond to different numbers, n_1 and n_2. The variances of the canonical variables $\hat{X}_j$ ($\hat{x}_j$ and $\hat{p}_j$) are defined as

$$V_{\mathrm{BG}}(X_j)={}_{\mathrm{BG}}\langle\beta;q|\hat{X}_j^2|\beta;q\rangle_{\mathrm{BG}}-({}_{\mathrm{BG}}\langle\beta;q|\hat{X}_j|\beta;q\rangle_{\mathrm{BG}})^2. \tag{3.15}$$

Recently, there is much interest in the so-called two-mode Schrödinger cat state $|\beta;q,\varphi\rangle$ suggested by Gerry and Grobe as an eigenstate of $\hat{K}_-^2$ [33]:

$$\hat{K}_-^2|\beta;q,\varphi\rangle=\beta^2|\beta;q,\varphi\rangle. \tag{3.16}$$

The execution of a little algebra with the above equation leads to

$$|\beta;q,\varphi\rangle=N_\varphi\left[\frac{|\beta|^q}{I_q(2|\beta|)}\right]^{1/2}\sum_{n=0}^{\infty}[1+(-1)^n e^{i\varphi}]\frac{\beta^n}{\sqrt{n!(q+n)!}}|n+q,q\rangle, \tag{3.17}$$

where

$$N_\varphi=\frac{1}{\sqrt{2}}\left[1+\frac{|\beta|^q}{I_q(2|\beta|)}\cos\varphi\sum_{n=0}^{\infty}\frac{(-1)^n|\beta|^{2n}}{n!(q+n)!}\right]^{-1/2}. \tag{3.18}$$

In fact, Schrödinger cat state is a superposition of two BG coherent states separated in phase by π [33]

$$|\beta;q,\varphi\rangle=N_\varphi[|\beta;q\rangle_{\mathrm{BG}}+e^{i\varphi}|-\beta;q\rangle_{\mathrm{BG}}]. \tag{3.19}$$

Various nonclassical properties of the Schrödinger cat states such as the sub-Poissonian quantum-number statistics and the violation of Cauchy-Schwarz inequality have been investigated in the literature [33, 34].

3.2 Perelomov Coherent State

Now we turn our attention to the investigation of the Perelomov coherent state which is an another useful type of SU(1, 1) coherent state. This state also involves tight correlations between the modes as in the case of BG coherent state [29] and can be expressed in terms of the number state as

$$\begin{aligned}|\xi;q\rangle_{\mathrm{P}}&=\exp(\beta\hat{K}_+-\beta^*\hat{K}_-)|q,0\rangle\\&=(1-|\xi|^2)^{(1+q)/2}\sum_{n=0}^{\infty}\left(\frac{(n+q)!}{n!q!}\right)^{1/2}\xi^n|n+q,n\rangle,\end{aligned} \tag{3.20}$$

where

$$\xi = \frac{\beta}{|\beta|} \tanh |\beta|. \tag{3.21}$$

The coordinate representation of (3.20) may be evaluated by using the formula [35]

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + it)^n \exp(-t^2) dt. \tag{3.22}$$

Thus, we have

$$\begin{aligned}
\langle x_1, x_2 | \xi; q \rangle_{\mathrm{P}} = {} & \sqrt{\frac{(\Omega_1 \Omega_2)^{1/2}}{\pi s_1 s_2 \hbar 2^q q!}} \left(\frac{1 - |\xi|^2}{1 - \xi^2} \right)^{(1+q)/2} e^{i p_{p,1}(t) x_1/\hbar} e^{i p_{p,2}(t) x_2/\hbar} \\
& \times H_q \left[\frac{1}{\sqrt{\hbar(1 - \xi^2)}} \left(\frac{\sqrt{\Omega_1}}{s_1} [x_1 - x_{p,1}(t)] - \frac{\sqrt{\Omega_2}}{s_2} \xi [x_2 - x_{p,2}(t)] \right) \right] \\
& \times \exp \Bigg\{ \frac{1}{2\hbar} \left(\frac{\Omega_1}{s_1^2} [x_1 - x_{p,1}(t)]^2 - \frac{\Omega_2}{s_2^2} [x_2 - x_{p,2}(t)]^2 \right) \\
& - \frac{1}{\hbar(1 - \xi^2)} \left[\frac{\sqrt{\Omega_1}}{s_1} [x_1 - x_{p,1}(t)] - \frac{\sqrt{\Omega_2}}{s_2} \xi [x_2 - x_{p,2}(t)] \right]^2 \\
& - \frac{i}{2\hbar} \sum_{j=1}^{2} \frac{B_j s_j - \dot{s}_j}{s_j A_j} [x_j - x_{p,j}(t)]^2 \Bigg\}.
\end{aligned} \tag{3.23}$$

In the limit of the standard harmonic oscillator, this recovers to that in [36]. The density operator in the Perelomov coherent state is

$$\hat{\rho}_{\mathrm{P}} = |\xi; q\rangle_{\mathrm{P}} \, {}_{\mathrm{P}}\langle \xi; q|. \tag{3.24}$$

The state of the system in the Hilbert space is completely represented by its density operator. The probability of finding $n + q$ quanta in mode 1 and n quanta in mode 2 is

$$P_n = \langle n + q, n | \hat{\rho}_{\mathrm{P}} | n + q, n \rangle = \frac{(q + n)!}{n! q!} |(1 - |\xi|^2)|^{1+q} |\xi|^{2n}. \tag{3.25}$$

When we evaluating the expectation values of the canonical variables and their square in this coherent states, it may useful to employ the integral formula [35]:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Gamma \left(n + \nu + \frac{1}{2} \right) t^{-2n-2\nu-1} = \Gamma \left(\nu + \frac{1}{2} \right) (t^2 - 1)^{-\nu-1/2}. \tag{3.26}$$

Hence, with the aid of this, we readily derive the representations

$${}_{\mathrm{P}}\langle \xi; q | \hat{x}_1 | \xi; q \rangle_{\mathrm{P}} = \sqrt{\frac{s_1^2 \hbar}{2\Omega_1}} \frac{(1 - |\xi|^2)^{1+q}}{q!} \sum_{n=0}^{\infty} \frac{(n + q + 1)!}{n! \sqrt{n + 1}} |\xi|^{2n} (\xi + \xi^*) + x_{p,1}(t), \tag{3.27}$$

$${}_{\mathrm{P}}\langle \xi; q | \hat{x}_2 | \xi; q \rangle_{\mathrm{P}} = \sqrt{\frac{s_2^2 \hbar}{2\Omega_2}} \frac{(1 - |\xi|^2)^{1+q}}{q!} \sum_{n=0}^{\infty} \frac{(n + q + 1)!}{n! \sqrt{n + q + 1}} |\xi|^{2n} (\xi + \xi^*) + x_{p,2}(t), \tag{3.28}$$

$$
{}_{\mathrm{P}}\langle \xi; q|\hat{p}_1|\xi; q\rangle_{\mathrm{P}} = -\sqrt{\frac{\hbar}{2}}\frac{(1-|\xi|^2)^{1+q}}{q!}\sum_{n=0}^{\infty}\frac{(n+q+1)!}{n!\sqrt{n+1}}|\xi|^{2n}(Y_1\xi + Y_1^*\xi^*) + p_{p,1}(t), \tag{3.29}
$$

$$
{}_{\mathrm{P}}\langle \xi; q|\hat{p}_2|\xi; q\rangle_{\mathrm{P}} = -\sqrt{\frac{\hbar}{2}}\frac{(1-|\xi|^2)^{1+q}}{q!}\sum_{n=0}^{\infty}\frac{(n+q+1)!}{n!\sqrt{n+q+1}}|\xi|^{2n}(Y_2\xi + Y_2^*\xi^*) + p_{p,2}(t), \tag{3.30}
$$

$$
\begin{aligned}
{}_{\mathrm{P}}\langle \xi; q|\hat{x}_1^2|\xi; q\rangle_{\mathrm{P}} = {}& \frac{(1-|\xi|^2)^{1+q}}{q!}\Bigg[\frac{s_1^2\hbar}{2\Omega_1}\sum_{n=0}^{\infty}\frac{(n+q+2)!}{n!\sqrt{(n+1)(n+2)}}|\xi|^{2n}(\xi^2+\xi^{*2}) \\
&+ x_{p,1}(t)\sqrt{\frac{2s_1^2\hbar}{\Omega_1}}\sum_{n=0}^{\infty}\frac{(n+q+1)!}{n!\sqrt{n+1}}|\xi|^{2n}(\xi+\xi^*)\Bigg] \\
&+ \frac{s_1^2\hbar}{2\Omega_1}\frac{1}{1-|\xi|^2}(2q+1+|\xi|^2) + x_{p,1}^2(t),
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
{}_{\mathrm{P}}\langle \xi; q|\hat{x}_2^2|\xi; q\rangle_{\mathrm{P}} = {}& \frac{(1-|\xi|^2)^{1+q}}{q!}\Bigg[\frac{s_2^2\hbar}{2\Omega_2}\sum_{n=0}^{\infty}\frac{(n+q+2)!}{n!\sqrt{(n+q+1)(n+q+2)}}|\xi|^{2n}(\xi^2+\xi^{*2}) \\
&+ x_{p,2}(t)\sqrt{\frac{2s_2^2\hbar}{\Omega_2}}\sum_{n=0}^{\infty}\frac{(n+q+1)!}{n!\sqrt{n+q+1}}|\xi|^{2n}(\xi+\xi^*)\Bigg] \\
&+ \frac{s_2^2\hbar}{2\Omega_2}\frac{1}{1-|\xi|^2}(1+(2q+1)|\xi|^2) + x_{p,2}^2(t),
\end{aligned} \tag{3.32}
$$

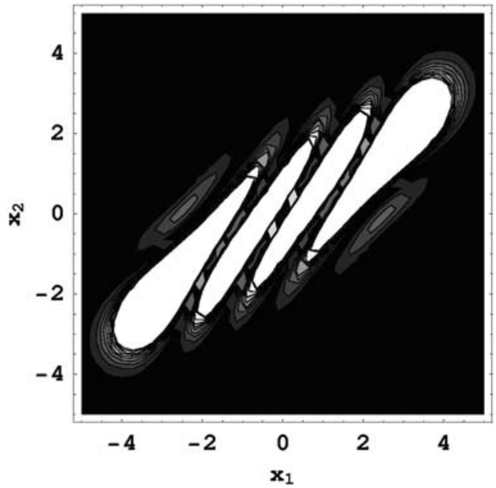
$$
\begin{aligned}
{}_{\mathrm{P}}\langle \xi; q|\hat{p}_1^2|\xi; q\rangle_{\mathrm{P}} = {}& \frac{(1-|\xi|^2)^{1+q}}{q!}\Bigg[\frac{\hbar}{2}\sum_{n=0}^{\infty}\frac{(n+q+2)!}{n!\sqrt{(n+1)(n+2)}}|\xi|^{2n}(Y_1^2\xi^2+Y_1^{*2}\xi^{*2}) \\
&- \sqrt{2\hbar}p_{p,1}(t)\sum_{n=0}^{\infty}\frac{(n+q+1)!}{n!\sqrt{n+1}}|\xi|^{2n}(Y_1\xi+Y_1^*\xi^*)\Bigg] \\
&+ \frac{\hbar}{2}|Y_1|^2\frac{1}{1-|\xi|^2}(2q+1+|\xi|^2) + p_{p,1}^2(t),
\end{aligned} \tag{3.33}
$$

$$
\begin{aligned}
{}_{\mathrm{P}}\langle \xi; q|\hat{p}_2^2|\xi; q\rangle_{\mathrm{P}} = {}& \frac{(1-|\xi|^2)^{1+q}}{q!}\Bigg[\frac{\hbar}{2}\sum_{n=0}^{\infty}\frac{(n+q+2)!}{n!\sqrt{(n+q+1)(n+q+2)}}|\xi|^{2n}(Y_2^2\xi^2+Y_2^{*2}\xi^{*2}) \\
&- \sqrt{2\hbar}p_{p,2}(t)\sum_{n=0}^{\infty}\frac{(n+q+1)!}{n!\sqrt{n+q+1}}|\xi|^{2n}(Y_2\xi+Y_2^*\xi^*)\Bigg] \\
&+ \frac{\hbar}{2}|Y_2|^2\frac{1}{1-|\xi|^2}(1+(2q+1)|\xi|^2) + p_{p,2}^2(t).
\end{aligned} \tag{3.34}
$$

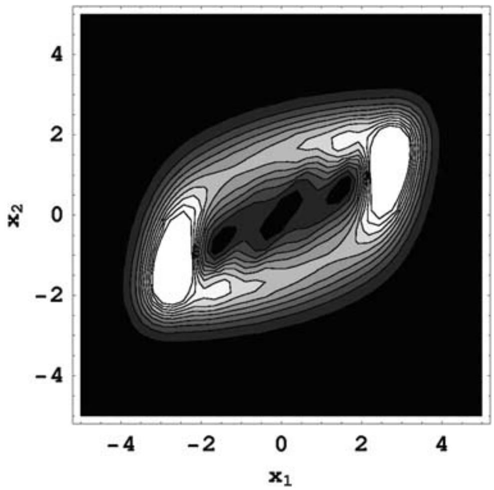
Similarly to the previous case, we can also identify the variance of canonical variables from

$$
V_{\mathrm{P}}(X_j) = {}_{\mathrm{P}}\langle \xi; q|\hat{X}_j^2|\xi; q\rangle_{\mathrm{P}} - ({}_{\mathrm{P}}\langle \xi; q|\hat{X}_j|\xi; q\rangle_{\mathrm{P}})^2. \tag{3.35}
$$

Fig. 1 Contour plots for the time evolution of the quadrature distribution of the probability density in BG coherent state, (3.2), for two-mode CK oscillator with no driving force ($f_j(t)=0$) when (**a**) $t=0$; (**b**) $t=0.5$; (**c**) $t=1.0$; and (**d**) $t=1.5$. We used $\Omega_1=\Omega_2=1$, $\omega_{0,1}=\omega_{0,2}=1$, $\gamma_1=\gamma_2=0.5$, $M_1=M_2=1$, $\hbar=1$, $x_{0,1}=x_{0,2}=2^{3/2}$, $q=3$, and $\phi_1=\phi_2=0$



(a)



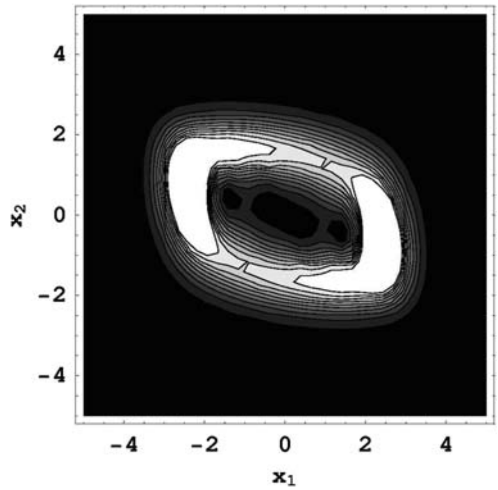
(b)

4 Application to the Two-Mode CK Oscillator

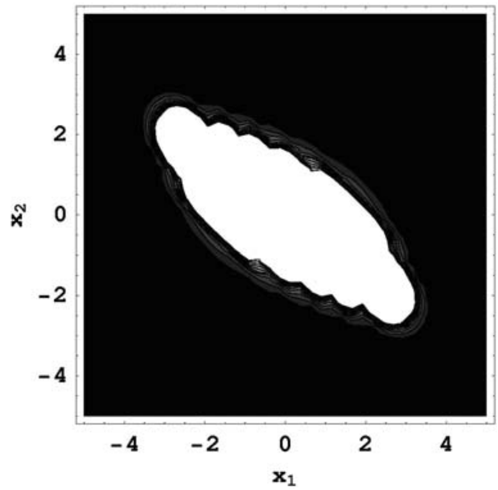
With the free choice of time functions $A_j(t) - F_j(t)$, our development in the previous sections may be employed diverse types of TDQHS. As an application of our development, let's consider a special case that described by two-mode CK Hamiltonian with the choice of

$$A_j = \frac{1}{M_j e^{\gamma_j t}}, \qquad C_j = M_j \omega_{0,j}^2 e^{\gamma_j t}, \qquad D_j = -M_j e^{\gamma_j t} f_j(t), \tag{4.1}$$

Fig. 1 (*Continued*)



(c)



(d)

where M_j, γ_j and $\omega_{0,j}$ are real positive constants and all other functions are zero: $B_j = E_j = F_j = 0$. Then, $\hat{H}_j(\hat{x}_j, \hat{p}_j, t)$ in (2.2) becomes

$$\hat{H}_j(\hat{x}_j, \hat{p}_j, t) = \frac{\hat{p}_j^2}{2M_j} e^{-\gamma_j t} + \frac{1}{2} M_j e^{\gamma_j t} [\omega_{0,j}^2 \hat{x}_j^2 - 2 f_j(t) \hat{x}_j], \tag{4.2}$$

In this case, the solutions of (2.9) are given by

$$s_j(t) = \sqrt{\frac{\Omega_j}{M_j}} \tilde{\omega}_j^{-1/2} \exp\left(-\frac{\gamma_j}{2} t\right), \tag{4.3}$$

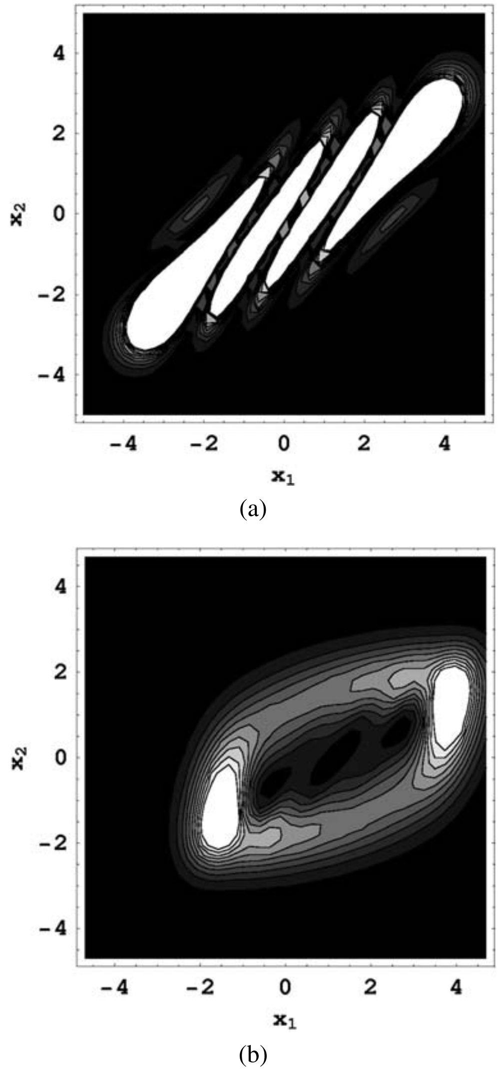


Fig. 2 Same as in Fig. 1, but with a driving force (4.5) in mode 1. We used $f_{0.1} = 1$, $\omega_{d,1} = 0.97$, and $\phi_{d,1} = 0$

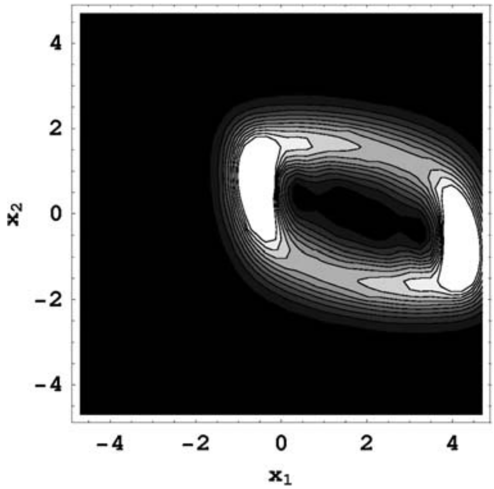
where $\tilde{\omega}_j$ are modified frequencies of the form

$$\tilde{\omega}_j = \left(\omega_{0,j}^2 - \frac{\gamma_j^2}{4} \right)^{1/2}. \tag{4.4}$$

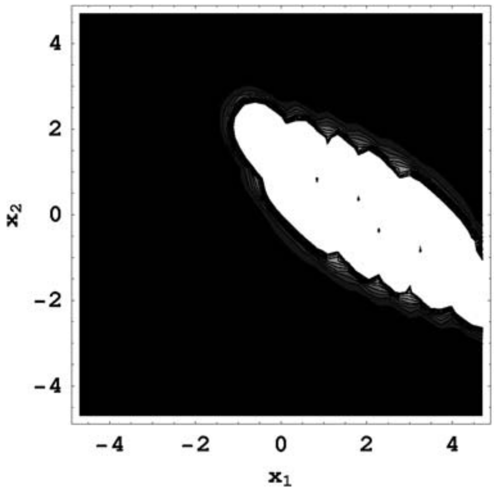
We choose the driving forces $f_j(t)$ as

$$f_j(t) = f_{0,j} \cos(\omega_{d,j} t + \phi_{d,j}), \tag{4.5}$$

Fig. 2 (*Continued*)



(c)



(d)

with real amplitudes $f_{0,j}$ and arbitrary phases $\phi_{d,j}$. Then, the complementary solutions and the particular solutions for mode j are written as

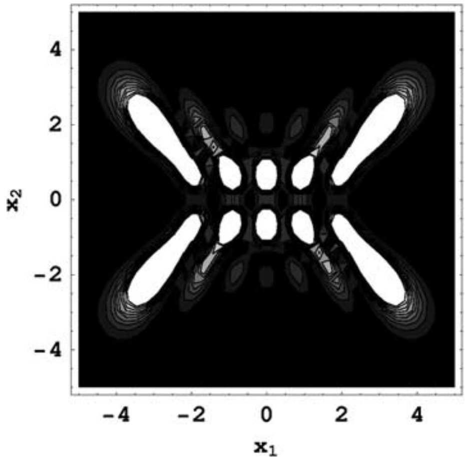
$$x_{c,j}(t) = x_{0,j} e^{-\gamma_j t/2} \cos(\tilde{\omega}_j t + \phi_j), \tag{4.6}$$

$$p_{c,j}(t) = -M_j x_{0,j} e^{\gamma_j t/2} \left[\frac{\gamma_j}{2} \cos(\tilde{\omega}_j t + \phi_j) + \tilde{\omega}_j \sin(\tilde{\omega}_j t + \phi_j) \right], \tag{4.7}$$

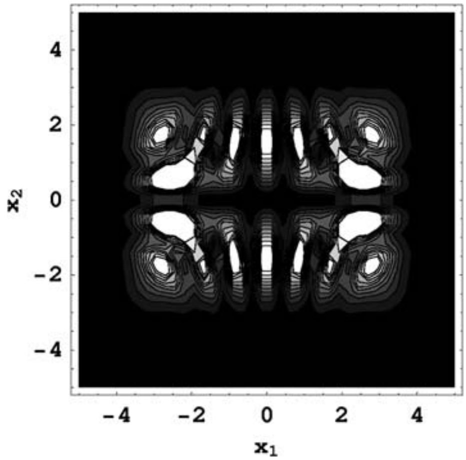
$$x_{p,j}(t) = \frac{f_{0,j}}{\sqrt{(\omega_{0,j}^2 - \omega_{d,j}^2)^2 + \gamma_j^2 \omega_{d,j}^2}} \cos(\omega_{d,j} t + \phi_{d,j} - \delta_j), \tag{4.8}$$

$$p_{p,j}(t) = -\frac{M_j f_{0,j} \omega_{d,j}}{\sqrt{(\omega_{0,j}^2 - \omega_{d,j}^2)^2 + \gamma_j^2 \omega_{d,j}^2}} e^{\gamma_j t} \sin(\omega_{d,j} t + \phi_{d,j} - \delta_j), \tag{4.9}$$

Fig. 3 Contour plots for the time evolution of the quadrature distribution of the probability density in Schrödinger cat state, (3.19), for two-mode CK oscillator with no driving force ($f_j(t)=0$) when (**a**) $t=0$; (**b**) $t=0.5$; (**c**) $t=1.0$; and (**d**) $t=1.5$. We used $\Omega_1=\Omega_2=1$, $\omega_{0,1}=\omega_{0,2}=1$, $\gamma_1=\gamma_2=0.5$, $M_1=M_2=1$, $\hbar=1$, $x_{0,1}=x_{0,2}=2^{3/2}$, $q=3$, $\phi_1=\phi_2=0$, and $\varphi=\pi$



(a)



(b)

where $x_{0,j}$ is amplitudes of oscillation at $t=0$, ϕ_j are initial phases and

$$\delta_j = \tan^{-1}\frac{\gamma_j\omega_{d,j}}{\omega_{0,j}^2-\omega_{d,j}^2}. \tag{4.10}$$

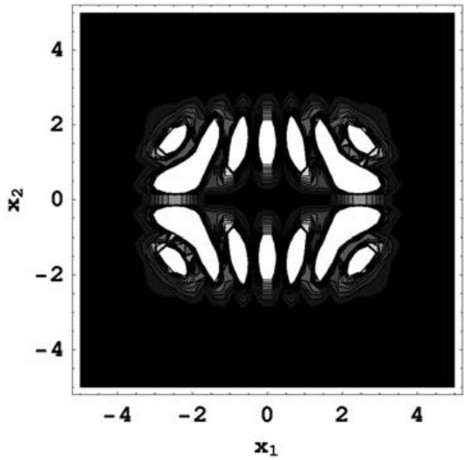
In terms of classical solutions given in (4.3) and (4.6)–(4.9), the system can be completely described. If we consider (2.13) and the third of (2.14), β in (3.1) becomes

$$\beta = \beta_1\beta_2, \tag{4.11}$$

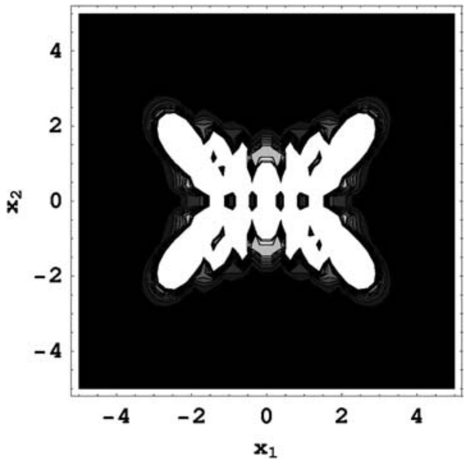
where

$$\beta_j = \sqrt{\frac{1}{2\hbar\Omega_j}}\left\{\left[\frac{\Omega_j}{s_j}+i\frac{B_js_j-\dot{s}_j}{A_j}\right]x_{c,j}(t)+is_jp_{c,j}(t)\right\}. \tag{4.12}$$

Fig. 3 (*Continued*)



(c)



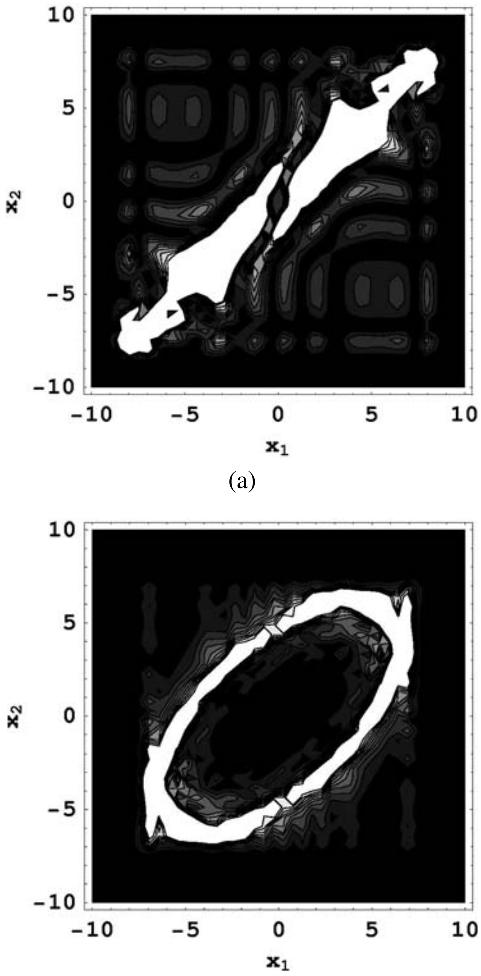
(d)

Then, we can evaluate β for two-mode CK oscillator by substituting (4.6) and (4.7) into the above equation. Thus, we have

$$\beta(t) = \beta_0 e^{-i[(\tilde{\omega}_1+\tilde{\omega}_2)t+\phi_1+\phi_2]}, \tag{4.13}$$

where $\beta_0 = \sqrt{M_1 M_2 \tilde{\omega}_1 \tilde{\omega}_2} x_{0,1} x_{0,2}/(2\hbar)$. We depicted contour plots of the probability density for the BG coherent state in Figs. 1 and 2, the Schrödinger cat state in Fig. 3, and the Perelomov coherent state in Fig. 4 for two-mode CK oscillator. One may compare these figures with those of two-mode simple harmonic oscillator case which are represented in [36]. All of the probability densities converge to the origin as time goes by owing to the dissipation of energy. The effect of driving force can be seen from Fig. 2. The probability density in Fig. 2 more or less deviates from the origin with time due to the influence of driving force.

Fig. 4 Contour plots for the time evolution of the quadrature distribution of the probability density in Perelomov coherent state, (3.23), for two-mode CK oscillator with no driving force ($f_j(t) = 0$) when (**a**) $t = 0$; (**b**) $t = 0.5$; (**c**) $t = 1.0$; and (**d**) $t = 1.5$. We used $\Omega_1 = \Omega_2 = 1$, $\omega_{0,1} = \omega_{0,2} = 1$, $\gamma_1 = \gamma_2 = 0.5$, $M_1 = M_2 = 1$, $\hbar = 1$, $x_{0,1} = x_{0,2} = 2^{3/2}$, $q = 3$, and $\phi_1 = \phi_2 = 0$



(a)

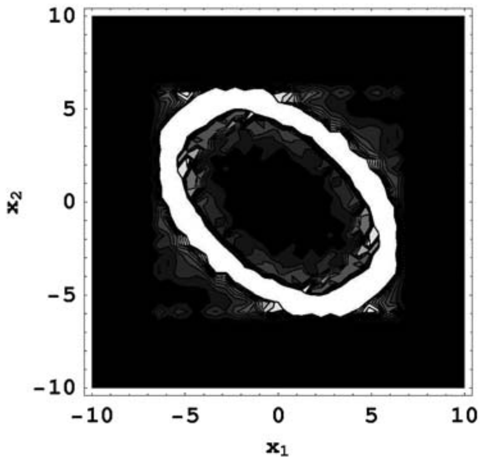
(b)

5 Conclusion

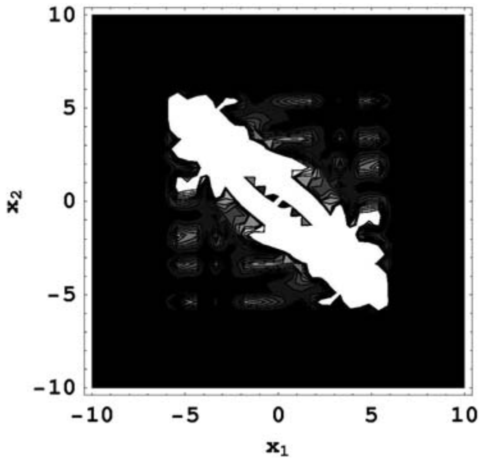
Two-mode bosonic realization of SU(1, 1) Lie algebra is very useful in describing the nondegenerate parametric amplifier while one mode realization is used to describe the degenerate parametric amplifier [14]. The quantum properties of two-mode TDQHS are investigated through SU(1, 1) Lie algebraic formulation. Two-mode SU(1, 1) generators, $\hat{K}_0$, $\hat{K}_1$, and $\hat{K}_2$, which satisfy conventional commutation relations between themselves are constructed in (2.6)–(2.8). In terms of $\hat{K}_1$, and $\hat{K}_2$, raising operator $\hat{K}_+$ and lowering operator $\hat{K}_-$ are defined as (2.11). We confirmed that $\hat{K}_0$ plays the role of invariant operator since its time-derivative vanishes. From (2.17) and (2.18), we confirm that $\hat{K}_+$ and $\hat{K}_-$ can be expressed in terms of time-constant magnitude and time-dependent phase factor.

Two kind of the generalized SU(1, 1) coherent states, i.e., BG coherent state and Perelomov coherent state are studied. These coherent states are represented in terms of the classical solutions for coordinate and momentum, i.e., complementary solutions $x_c(t)$ and $p_c(t)$ plus a particular solutions $x_p(t)$ and $p_p(t)$. The wave functions in SU(1, 1) coherent states can be

Fig. 4 (*Continued*)



(c)



(d)

used to study various quantum properties of the system. Exact wave function in Perelomov coherent state is derived using SU(1, 1) Lie algebraic approach. The expectation values of coordinates and momenta and their squares are calculated in both BG coherent state and Perelomov coherent state.

As an application, we have employed our development to study SU(1, 1) coherent states of two-mode CK oscillator. From (4.13), we can easily find that the magnitude of eigenvalue $\beta(t)$ for the lowering operator is constant with time. This is closely related to the fact that the magnitude of $\hat{K}_-$ does not vary with time (see (2.18)). The time-behavior of the probability density in both BG and Perelomov coherent states for the two-mode CK oscillator are shown in figures. From these figures, we see that the probability density gradually converges to the origin as time goes by on account of the dissipation of energy. The effect of driving force can be confirmed from Fig. 2. According to this figure, the center of probability density have been deviated from the origin due to the effect of driving force.

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